Problem 14

Suppose f is a function with the property that $|f(x)| \leq x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.

Solution

Suppose that $|f(x)| \leq x^2$ for all x. Set $x = 0$.

$$|f(0)| \leq 0^2$$

$$|f(0)| \leq 0$$

$$-0 \leq f(0) \leq 0$$

$$0 \leq f(0) \leq 0$$

$$f(0) = 0$$

Now consider the derivative of f at $x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h}$$

Take the absolute value of both sides.

$$|f'(0)| = \left| \lim_{h \to 0} \frac{f(h)}{h} \right|$$

The absolute value function is continuous, so the limit may be brought outside.

$$|f'(0)| = \lim_{h \to 0} \left| \frac{f(h)}{h} \right|$$

$$= \lim_{h \to 0} \frac{|f(h)|}{|h|}$$

Use the fact that $|f(h)| \leq h^2$.

$$|f'(0)| \leq \lim_{h \to 0} \frac{h^2}{|h|}$$

$$= \lim_{h \to 0} \frac{|h||h|}{|h|}$$

$$= \lim_{h \to 0} |h|$$

$$= 0$$

Therefore,

$$|f'(0)| \leq 0$$

$$-0 \leq f'(0) \leq 0$$

$$0 \leq f'(0) \leq 0$$

$$f'(0) = 0.$$